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New progress on $\mathfrak{L}(E, F) = N_1(E, F)$

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This paper is devoted to the following problem posed by A. Grothendieck:

Let E and F be two Banach spaces such that every operator from E into F is nuclear. Is E or F finite dimensional?

We shall show that the answer is yes provided that E or F is of the type $\mathcal{G}$.

Definition. A Banach space E is said to be of the type $\mathcal{G}$ provided that if every composition

$$(*) \qquad c_0 \xrightarrow{t} E' \xrightarrow{s} l_2 \xrightarrow{j} c_0$$

is an absolutely summing operator then E is either finite dimensional or E contains uniformly the spaces $l_\infty(n)$ (equivalently, E' contains uniformly complemented subspaces F_n with $\sup d(F_n, l_1(n)) \leqslant \infty$). Here j denotes the canonical inclusion.

The notations and terminology is standard.

If every composition (*) defines an absolutely summing operator then there exists an $M > 0$ so that $\pi_1(j \cdot S \cdot T) \leqslant M\|S\| \cdot \|T\|$ for all $S \in (E', l_2)$ and $T \in \mathfrak{L}(c_0, E')$ and in such a case E' cannot contain for $p = 2$ or $p = \infty$ sequences of uniformly complemented subspaces F_n with $\sup d(F_n, l_p(n)) < \infty$.

Then every Banach space E which contains a sequence of λ-complemented subspaces E_n with $d(E_n, l_p(n)) \leqslant \lambda$ for some $p \in \{1, 2, \infty\}$ is of the type $\mathcal{G}$. According to the main result in [3] and Prop. 2.6 in [5] it follows that every Banach space E whose dual is complemented in a Banach lattice (E has local unconditional structure in the sense of Gordon and Lewis) is also of the type $\mathcal{G}$.

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REMARK. Let E be a Banach space. Then every operator $T \in \mathfrak{L}(E', l_2)$ is the pointwise limit of a net S'_α with $S_\alpha \in \mathfrak{L}(l_2, E)$ and $\|S_\alpha\| \leqslant \|T\|$. Moreover

$$\pi_1(T) = \lim \pi_1(S'_\alpha).$$

Proof. In fact, if $\{e_n\}_n$ denotes the canonical basis of l_2, we can produce the operators S_α simply by defining $S(\cdot) = \Sigma < e_n, \cdot > T^*(e_n)$ for each finite subset $\alpha \subset N$.

Lemma. An infinite Banach space E of the type $\mathcal{G}$ contains uniformly the spaces $l_\infty(n)$ provided that

$$(T \circ S \circ j')' \in \prod_1(l_\infty, l_\infty)$$

for every $T \in \mathfrak{L}(E, l_1)$ and $S \in \mathfrak{L}(l_2, E)$.

Proof. Notice that

$$j \circ R' \circ U \in \prod_1(c_0, c_0)$$

for every $U \in \mathfrak{L}(c_0, E')$ and every $R \in \mathfrak{L}(l_2, E)$. In fact, $(j \circ R' \circ U)' = (U' \circ Q) \circ \circ R \circ j'$ and thus $(j \circ R' \circ U)'' \in \prod_1(l_\infty, l_\infty)$. Here Q denotes the canonical embedding of E into E''. By the Closed Graph Theorem we check the existence of a positive $M > 0$ such that

$$\pi_1(j \circ R' \circ U) \leqslant M\|R\| \cdot \|U\|$$

for every $U \in \mathfrak{L}(c_0, E')$ and every $R \in \mathfrak{L}(l_2, E)$. According to our Remark above this implies that

$$\pi_1(j \circ S \circ T) \leqslant M\|S\| \cdot \|T\|$$

for every $T \in \mathfrak{L}(c_0, E')$ and every $S \in \mathfrak{L}(E', l_2)$ and our result follows.

We can now prove the following:

Theorem A. Let E, F be two infinite Banach spaces and let E be of the type $\mathcal{G}$.

i) If $\mathfrak{L}(E, F) = \prod_1(E, F)$ then E' contains uniformly the spaces $l_\infty(n)$ and F is isomorphic to a Hilbert space.

ii) If $T \in \mathfrak{L}(F, E)$ implies $T' \in \prod_1(E', F')$ then E contains uniformly the spaces $l_\infty(n)$ and F is isomorphic to a Hilbert space.

We shall prove only ii). First notice the existence of a constant $M > 0$ such that $\pi_1(T') \leqslant M\|T\|$ for all $T \in \mathfrak{L}(F, E)$. Dvoretzky's theorem on almost sferical sections of convex bodies asserts that for every $\varepsilon > 0$ and every $n \in N$ there exists a closed subspace $G \subset F$ of codimension n and an isomorphism $S : l_2(n) \to F/G$ such that $\|S\| \cdot \|S^{-1}\| \leqslant 1 + \varepsilon$. If $R \in \mathfrak{L}(l_2(n), E)$ and $\varphi : F \to F/G$ is the canonical mapping then $\pi_1((R \circ S^{-1})') = \pi_1((R \circ S^{-1} \circ \varphi)') \leqslant M\|R \circ S^{-1} \circ \varphi\| \leqslant M\|R\| \cdot \|S^{-1}\|$ and thus

$$\pi_1(R') = \pi_1((R \circ S^{-1} \circ S)') \leqslant \pi_1((R \circ S^{-1})')\|S\| \leqslant M(1 + \varepsilon)\|R\|.$$

For $T \in \mathfrak{L}(l_2, E)$ put $R_n = T/l_2(n)$ and let P_n be the canonical projection of l_2 onto $l_2(n)$. Then $Tx = \lim R_n \circ P_n(x)$ for all $x \in l_2$ and thus our Remark above implies that $\pi_1(T') = \lim \pi_1((R_n \circ P_n)') \leqslant M(1+\varepsilon)\|T\|$ so that by Lemma above E contains uniformly the spaces $l_\infty(n)$.

Then there exists a constant $C > 0$ such that $\pi_1(T) \leqslant C\|T\|$ for all $T \in \mathfrak{L}(l_1(n), F')$, $n \in N$. Now choose an onto mapping $\varphi \in \mathfrak{L}\ (l_1(\Gamma), F')$ for Γ a suitable index set. Then φ is absolutely summing and φ can be factored through a Hilbert space. Therefore F is isomorphic to a Hilbert space.

Theorem B. Let E and F be two Banach spaces and let E or F be of the type $\mathcal{G}$. If every operator from E into F is absolutely summing and hypermajorizing (i.e., the adjoint is absolutely summing) then E or F is finite dimensional.

Proof. Suppose that E and F are infinite dimensional.

If E is of the type $\mathcal{G}$ then, by Theorem A (i) the canonical inclusions $l_1(n) \to l_2(n)$ can be extended to operators $T_n \in \mathfrak{L}(E, F)$ with $\sup \|T_n\| < \infty$, in contradiction with the fact that the canonical inclusion $j' : l_1 \to l_2$ is not hypermajorizing. Then E and F cannot both be infinite dimensional.

If F is of the type $\mathcal{G}$ then by Theorem A(ii) E is isomorphic to a Hilbert space and F contains a sequence of uniformly complemented subspaces F_n such that $\sup d(F_n, l_\infty(n)) < \infty$. Then there exists a constant $M > 0$ (which does not depend of n) such that $\pi_1(T) \leqslant M\|T\|$ for every $T \in \mathfrak{L}(l_2(n), l_\infty(n))$, contradiction.

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